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Universality of low-energy scattering in three-dimensional field theory

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The universal low-energy behavior, $2mc/\ln|s-4m^2|$ of the scattering function of particles of positive mass m near the threshold $s=4m^2$, and $\pi/\ln|s-4m^2|$ for the corresponding S-wave phase shift, is established for weakly coupled field theory models with a mass $m>0$ in space-time dimension 3; c is a numerical constant independent of the model and couplings. This result is a nonperturbative property based on an exact analysis of the scattering function in terms of a two-particle irreducible (or Bethe-Salpeter) structure function. It also appears as generic by the same analysis in the framework of general relativistic quantum field theory. [S0556-2821(99)50208-7]

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There has been great interest within the last two decades in two-dimensional and, more recently, three-dimensional field theories both in view of their conceptual importance and for possible physical applications. In this Rapid Communication, we are more specifically concerned with the low-energy behavior of the (connected) two-body scattering function T in massive field theories in space-time dimension $d=2+1$; for definiteness, we consider models with one basic physical mass m associated with an elementary particle of the theory. The models rigorously defined so far for $d=3$ include $\lambda\phi^4$ and $\lambda\phi^4+\lambda'\phi^3$ at weak couplings [1]. By low-energy behavior of T, we mean its dominant behavior near the two-particle threshold $s=4m^2$, which is the lowest physical value of the squared center-of-mass energy s of the $2\rightarrow 2$ particle process considered. The aim of this Rapid Communication is (1) to show that, for all the previous field models, T enjoys a universal low-energy behavior equal to $2mc/\ln|s-4m^2|$ (a more complete analysis will be given in [2]), and (2) to indicate why such a universal behavior is generic in the class of all possible 3d field theories.

In fact, this universal behavior will appear as an intrinsically nonperturbative phenomenon, *spontaneously switched-on by the interaction* in $(2+1)$ dimensions.

In order to situate the history of the subject, let us recall the following fact. Starting from the basic principles of relativistic quantum field theory (QFT) together with unitarity, the following result concerning the partial waves $f_l(s)$ of the scattering function T (defined by $f_l(s)=1/\pi\int_0^{2\pi}T(s,\cos\theta)\cos l\theta\, d\theta$) was proven[1] in [3]:

$$f_l(s)=cs^{1/2}[\ln\sigma+b_l(s)]^{-1};\qquad(1)$$

in the latter, $\sigma=4m^2-s$, b_l is real analytic in a neighborhood of $s=4m^2$ apart from a possible (simple or multiple) pole at $s=4m^2$, and c is a well-specified constant depending on the normalization conventions, but *not on the theory*.

Considering more specifically the S-wave f_0, the following consequence of Eq. (1) has recently been pointed out by Chadan, Khuri, Martin, and Wu [7]: (a) either b_0 has no pole at $s=4m^2$, i.e., is locally analytic and thus bounded, in which case f_0 behaves near $s=4m^2$ as $2mc/\ln|\sigma|$ which is a ''universal behavior independent of the theory'' correspondingly, the phase shift δ_0 given by $\hat{f}_0=e^{i\delta_0}\sin\delta_0=(\pi/c\sqrt{s})f_0$ behaves as $\pi/\ln|\sigma|$, (b) or b_0 has a pole, e.g., in $1/\sigma$, in which case f_0 and δ_0 behave instead as cst σ near $s=4m^2$; the constants cst depend here on the theory.

We therefore see that for f_0 the derivation of the announced ''universality property'' is directly linked to proving that case (a) is satisfied[2] by all the field models considered and is actually generic in the framework of QFT. The claim that $b_0(s)$ should have no pole at $s=4m^2$ is highly nonobvious and nontrivial. On the contrary for $l\geqslant 1$ all b_l do have poles, in accord with the fact (explained later) that all the partial waves f_l, $l\geqslant 1$ and also the function $T(s,\cos\theta)-f_0(s)\equiv 2\Sigma_{l\geqslant 1}f_l(s)\cos l\theta$ are bounded by cst$|\sigma|$ near the threshold. The proof of the common universal behavior of T and f_0 therefore relies on a precise control of $b_0(s)$. This control will be achieved through an analysis of field-theoretical structure functions *which goes beyond perturbation theory*. The latter is in fact deeply misleading because all Feynman functions associated with two-particle-reducible diagrams (as those of the toy-structure below) exhibit divergences at threshold in powers of $\ln\sigma$ (see [3,7]).

Our method is based on the existence of an *exact Bethe-Salpeter equation*, namely

[1]Equation (1) is the special case $d=3$ of Eq. (7) of [3b]. More precisely, this result follows [see in [3b] the argument between Eqs. (4) and (7), completed by the note (14)] from local physical-sheet analyticity, proved [4] from the standard Lehman-Symanzik-Zimmermann (L.S.Z.) or Wightman axioms of relativistic quantum field theory [5], unitarity (written for the partial waves) and a regularity assumption (e.g. continuity) on T at $s>4m^2$ in order to avoid a la Martin pathologies [6].

[2]In [7], a rather complete analysis of the same structure is given in the framework of nonrelativistic scattering theory; as an extension of earlier results of [8], it is found that the universal low-energy behavior $\pi/\ln|k|$ of δ_0 (k being the nonrelativistic analogue of σ) actually holds for a large class of potentials.

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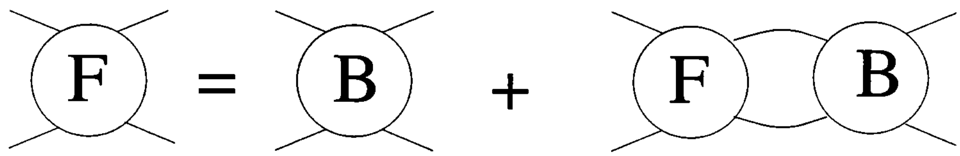
$$F(K;k,k') = B(K;k,k') + \int_{\Gamma(K)} \frac{F(K;k,\mathrm{k})B(K;\mathrm{k},k')}{[(K/2+\mathrm{k})^2-m^2][(K/2-\mathrm{k})^2-m^2]} d^3\mathrm{k} \tag{2}$$

which allows one to compute the complete four-point function F of the field, namely the off-shell extrapolation of the scattering function T, in terms of a well-defined ''two-particle-irreducible structure function'' (or ''Bethe-Salpeter kernel'') called B. For simplicity, we have written Eq. (2) in such a way that, in their definitions, B and F include as internal factors the pair of amputated two-point functions of the incoming energy-momenta normalized to 1 on the mass-shell.[3] In Eq. (2), K represents the total energy-momentum vector of the channel considered ($K^2=s$), while k and k' denote correspondingly the incoming and outgoing relative energy-momenta of this channel; these vectors can be complex and the (Feynman-type) integral in Eq. (2) is taken on a well-specified cycle $\Gamma(K)$ in complex k space. A usual suggestive graphical notation (of generalized Feynman type) for Eq. (2) is



(2′)

in which the internal lines of the one-loop diagram carry, respectively, the energy-momentum vectors $\mathrm{k}_1=K/2+\mathrm{k}$ and $\mathrm{k}_2=K/2-\mathrm{k}$. The mass shell, defined by $\mathrm{k}_1^2=\mathrm{k}_2^2=m^2$, is (for fixed K) a sphere of radius σ; at threshold, the vanishing of this sphere is responsible for the fact that *the cycle* $\Gamma(K)$ *becomes pinched by the pair of poles in the integral of Eq. (2)*. In a shorter notation, we shall also rewrite Eqs. (2) or (2′) as $F=B+F\circ B$, where $\circ$ represents the Feynman-type integration of Eq. (2) including two poles.

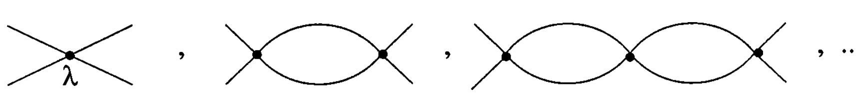
In perturbation theory, Eq. (2′) just represents a diagram counting identity in which the kernel B is the (formal) sum of four-point Feynman functions corresponding to all diagrams which are *two-particle-irreducible* in the scattering channel considered: *only these perturbative functions have no divergences at threshold.*

The fact that B does exist in the exact field theories as an analytic function related to F by the Fredholm-type equation (2) has been rigorously established in our past works [9], based on the pioneering ideas of Symanzik [10]. More precisely, two sets of results hold and are used here.

(i) Results implied by the basic principles of QFT together with ''off-shell unitarity'' (i.e., ''asymptotic completeness of the fields'') and regularity assumptions; they state that (a) B exists and is analytic in a region of the s-plane ($s=K^2$) containing the threshold $4m^2$, in particular, it has to be *uniform* around the threshold [9a,b] and (b) in view of Eq. (2), the corresponding singular structure of F at threshold is a pure consequence of the pinching of the cycle $\Gamma(K)$ by the two poles of the integration operation $\circ$, mentioned above [9a,c].

(ii) Results of constructive field theory: for all small coupling models that have been constructed for $d=3$ (and $d=2$), the function B can be rigorously defined and controlled in terms of the couplings; the property (ia) of B is then built-in, together with the exact Bethe-Salpeter equation (2) (see [11] and our comments below).

In order to make the ideas of our proof more transparent, it is very illustrative to restrict our attention to the $\lambda\phi^4$ model and to a ''toy Bethe-Salpeter structure'' already exhibiting all the features of the exact Bethe-Salpeter structure. This toy structure is obtained by considering the Feynman functions associated with the following bubble diagrams



[3]This amounts to making a ''wave-function normalization'' which fixes the constant c in Eq. (1) via unitarity.

If $B=B_0^{(\lambda)}\equiv\lambda|1\rangle\langle 1|$ denotes the constant kernel equal to λ, represented by the single-vertex diagram in the previous picture, all these Feynman functions are the successive iterated terms $B_0^{(\lambda)}\circ B_0^{(\lambda)}\cdots\circ B_0^{(\lambda)}$ (or Neumann series) of a Bethe-Salpeter equation $F_0^{(\lambda)}=B_0^{(\lambda)}+F_0^{(\lambda)}\circ B_0^{(\lambda)}$, which can be solved by elementary algebra. In fact, $F_0^{(\lambda)}$ is, like $B_0^{(\lambda)}$, a separable (or rank one) kernel given by the formula

$$F_0^{(\lambda)}(s)=\frac{\lambda|1\rangle\langle 1|}{1-\lambda\langle 1|\circ|1\rangle(s)}, \tag{3}$$

$$\text{where}\quad \langle 1|\circ|1\rangle(K^2)=\cdots$$

$$\int \frac{1}{[(K/2+\mathrm{k})^2-m^2][(K/2-\mathrm{k})^2-m^2]}d^3\mathrm{k}=\cdots$$

$$i\pi^2\ s^{-1/2}\ln\sigma+h(s),\quad h(s)\ \text{analytic at}\ s=4m^2; \tag{4}$$

$\langle 1|\circ|1\rangle(K^2)$ is, up to the factor λ^2, the one-bubble Feynman function of the previous picture. The perturbative expansion of $F_0^{(\lambda)}$, equal to the geometrical series

$$\sum_p \lambda^{p+1}[\langle 1|\circ|1\rangle(s)]^p=\sum_p \lambda^{p+1}[i\pi^2 s^{-1/2}\ln\sigma+h(s)]^p,$$

is of course completely misleading since at fixed λ its general (p bubble) term diverges like $(\ln\sigma)^p$ at threshold, while in view of Eqs. (3) and (4), the function $F_0^{(\lambda)}$ is in fact a unitary amplitude of the form (1), with $b_o(s)=cs^{1/2}[h(s)-1/\lambda]$, $c=-i\pi^{-2}$. For any λ different from zero, $b_0(s)$ is bounded [like $h(s)$] and therefore $F_0^{(\lambda)}$ has the universal behavior $2mc/\ln\sigma$ near $\sigma=0$.

Coming back to the exact Bethe-Salpeter equation of any given three-dimensional field theory with four-point function F and associated scattering function T, we will now show that the previous toy-structure and related phenomenons at threshold govern basically the corresponding behavior of the S-wave f_0 of T. It is convenient to use the *S-wave part F_0 of the four-point function F*, whose definition in terms of F is similar to that of f_0 in terms of T except that it is applied to off-shell Euclidean configurations and then analytically continued in the domain implied by the basic principles of QFT. F_0 is an off-shell extrapolation of f_0 which satisfies a partial-wave Bethe-Salpeter equation $F_0=B_0+F_0\circ B_0$ in terms of the S-wave part B_0 of B [12]. The particularity of the kernels $B_0,F_0,\cdots$ with respect to $B,F,\cdots$ is that they only depend on the Lorentz invariant variables $K^2=s,k_i^2=(K/2\pm k)^2,k_i'^2=(K/2\pm k')^2,i=1,2,$ and not on the transfer $t=(k-k')^2$. The kernels $B_0^{(\lambda)}$ and $F_0^{(\lambda)}$ of the previous toy-structure are of this nature.

Our argument is based on the following appropriate splitting $B_0=B_0'+B_0''$ of the given kernel $B_0(s;k_1^2,k_2^2;k_1'^2,k_2'^2)$: (a) $B_0'=B_0(s;k_1^2,k_2^2;m^2,m^2)$ is a separable (or rank one) kernel which we write (with the previous bra-ket notation) $B_0'=|\Psi_0\rangle\langle 1|$; (b) correspondingly the kernel $B_0''(s;k_1^2,k_2^2;k_1'^2,k_2'^2)$ satisfies the property that $B_0''(s;k_1^2,k_2^2;m^2,m^2)\equiv 0$. The same property is of course shared by all the iterated kernels $B_0''\circ B_0''\cdots\circ B_0''$ and by the corresponding solution F_0'' of the auxiliary Bethe-Salpeter equation $F_0''=B_0''+F_0''\circ B_0''$; moreover (this is less trivial, but crucial), all of them are also analytic in s at $s=4m^2$ up to the ''nongeneric case'' of the Fredholm alternative which would produce a pole for F_0''; in other words *no* $\ln\sigma$ *singularity is present in these kernels*. The reason for it is that B_0'' can be written as a sum of two terms which, respectively, factor out $k'^2_i-m^2,i=1,2,$ and therefore cancel out either one of the poles in the $\circ$ operation; as an effect, the cycle $\Gamma(K)$ is no more pinched at $K^2=4m^2$ and analyticity at threshold is preserved generically in solving this auxiliary Bethe-Salpeter equation.

The splitting of B_0 implies the following relation between the Fredholm resolvents F_0,F_0'' of B_0 and B_0'': $\mathbf{1}+F_0=(\mathbf{1}+F_0'')\circ[\mathbf{1}-B_0'\circ(\mathbf{1}+F_0'')]^{-1}$. Taking then into account the special form of B_0' and the fact that F_0'' vanishes for $k'^2_1=k'^2_2=m^2$ yields the following exact formula for the S-wave $f_0(s)=F_0(s;m^2,m^2;m^2,m^2)$ of the theory considered:

$$f_0(s)=\frac{(|\Psi\rangle\langle 1|)(s;k_1^2,k_2^2)_{|k_1^2=k_2^2=m^2}}{1-\langle 1|\circ|\Psi\rangle(s)}, \tag{5}$$

where:

$$|\Psi\rangle=|\Psi_0\rangle+F_0''\circ|\Psi_0\rangle. \tag{6}$$

Equation (5) is of the same form as Eq. (3), to which it reduces in an obvious way when $B_0=B_0'=B_0^{(\lambda)}$, i.e., when $|\Psi_0\rangle=\lambda|1\rangle$ and $B_0''=F_0''=0$. However, the remarkable fact is that Eqs. (5),(6) are exact (nonperturbative) equations, valid for the S-wave of any field theory. We shall now show that Eq. (5) exhibits a structure which is exactly of the form (1) with $b_0(s)$ bounded in ''generic cases'' and in particular for all weakly coupled models.

We notice that the function $\Psi(s;k_1^2,k_2^2)$ represented by $|\Psi\rangle$ (or by the kernel $|\Psi\rangle\langle 1|$) is generically analytic at $s=4m^2$: this follows from Eq. (6) by applying to F_0'' the ''nonpinching argument'' given above in (b). Let us now call $g(s)$ the numerator at the right-hand side of Eq. (5), namely $g(s)=\Psi(s;m^2,m^2)$. Then we claim that one can write

$$\langle 1|\circ|\Psi\rangle(s)=\langle 1|\circ|1\rangle(s)\times g(s)+l(s), \tag{7}$$

where $l(s)$ is generically analytic at $s=4m^2$. This follows from writing $\Psi(s;k_1^2,k_2^2)=g(s)+\Psi_{(reg)}(s;k_1^2,k_2^2)$, where $\Psi_{(reg)}$ vanishes at $k_1^2=k_2^2=m^2$ and therefore produces a function $l(s)=\langle 1|\circ|\Psi_{(reg)}\rangle$ regular at threshold [again in view of the nonpinching argument of (b), but used on the left]. In view of Eq. (7) [and by taking Eq. (4) into account], we can thus rewrite Eq. (5) as follows:

$$f_0(s)=\frac{g(s)}{1-l(s)-g(s)[c^{-1}s^{-1/2}\ln\sigma+h(s)]}. \tag{8}$$

We then conclude that f_0 is of the form (1), with

$$b_0(s)=cs^{1/2}\left(h(s)+\frac{l(s)-1}{g(s)}\right). \tag{9}$$

Since h is analytic, the question of the universality of f_0 in $(\ln\sigma)^{-1}$ amounts to discussing the generic character of the fact that $[l(s)-1]/g(s)$ has no pole at $s=4m^2$. The situation is as follows:

(i) In all weakly coupled models containing a $\lambda\phi^4$ term, it is claimed (see our comment below) that B is of the form $B=B_0^{(\lambda)}+O(\lambda^2)$. This entails that $\Psi_0=\lambda+O(\lambda^2)[s;k_1^2,k_2^2]$, while B_0'' and therefore F_0'' are bounded analytic functions of $s,k_1^2,k_2^2,k_1'^2,k_2'^2$ of order $O(\lambda^2)$. It then follows from Eq. (6) that $\Psi=\lambda+O(\lambda^2)[s;k_1^2,k_2^2]$ and therefore $g(s)=\lambda+O(\lambda^2)[s]$, while the analytic functions $\Psi_{(reg)}(s;k_1^2,k_2^2)$ and therefore $l(s)$ are of order $O(\lambda^2)$. One thus concludes that

$$\frac{l(s)-1}{g(s)}=\frac{O(\lambda^2)[s]-1}{\lambda+O(\lambda^2)[s]}, \tag{10}$$

which behaves like $-1/\lambda$ and is therefore *finite* in some range of couplings $(0<\lambda<\lambda_0)$: for this class of theories the universal behavior of f_0 at threshold is thus established; moreover, one can see as a by-product that f_0 exactly behaves as the function $F_0^{(\lambda)}$ of the toy-structure in the limit of small λ's.

(ii) For more general field theories, the universal behavior of f_0 is valid except if either $g(s)=0$ or $l(s)=\infty$, the latter case being produced by the Fredholm alternative in the auxiliary Bethe-Salpeter equation (i.e., $F_0''=\infty$). These exceptions are defined by the vanishing of analytic functions which in view of (i) cannot be identically zero (at least under the usual postulate of analytic continuation in the couplings). Such cases necessitate that B_0'' is large, i.e., that B_0 has a large rate of variation with respect to the masses near the mass-shell.

A similar analysis can be made for T (including in the models the dependence of T with respect to the couplings near $s=4m^2$) [2]. However, the fact that T and f_o enjoy the same universal behavior at threshold (with the possible exceptions analyzed above) is implied by the following general property: $T-f_0$ is bounded at small σ by cst$|\sigma|$. This property relies on the analyticity of $\hat{T}(s,t)\equiv T(s,\cos\theta)$ in a *fixed* neighborhood of $t=0$ in the complex plane of the variable $t=[(4m^2-s)/2](1-\cos\theta)$, for all s in a cut-neighborhood of $4m^2$, which is a rigorous result [4] of general QFT (under a regularity assumption of the type specified in [6], automatically satisfied in the models). The latter property implies that, if one rewrites $T-f_0=\Sigma_{l\geq 1}f_l(s)(z^l+z^{-l})$ with $z=e^{i\theta}$, this series in z has to converge in a ring of the form $C|\sigma|<|z|<(C|\sigma|)^{-1}$, where it is uniformly bounded in σ. By a standard argument of complex analysis, one concludes that each $f_l(s)$ is bounded at small σ by cst$(C|\sigma|)^l$, which then entails the announced bound on $T-f_0$. (These bounds on the f_l imply that for $l\geq 1$, each uniform function b_l of Eq. (1) does have a pole *of order* $\geq l$.)

Results of constructive QFT for weakly coupled 3D field models. The above-mentioned results on B can be obtained by combining a rigorous definition of 3D models of the type given for the model $\lambda\phi^4$ in [1] with a definition of B of the type given in [11a] for two-dimensional models, which is easily adaptable to the 3D case. They can also be derived as a specially simple case through the general methods given in [11b] for treating nonsuper-renormalizable models (such as the massive Gross-Neveu model in dimension 2). In these methods of constructive QFT, one defines the Fourier transforms $\tilde{F}$ and $\tilde{B}$ of F and B in Euclidean space-time through a certain type of expansions which are convergent at small couplings, in contrast to perturbative expansions. Diagrammatical analysis provides Eq. (2) [or Eq. (2′)]. On the other hand, expansions of $\tilde{B}$ only involve two-particle irreducible diagrams and this fact implies that $\tilde{B}$ satisfies better exponential falloff properties than $\tilde{F}$ (in Euclidean space-time). This implies in turn that B is analytic in a larger strip around Euclidean energy-momentum space, which includes the threshold $s=4m^2$. The analysis distinguishes the single vertex part equal to the constant λ from a remainder (i.e., the sum of all the other two-particle irreducible contributions) that is uniformly bounded by cstλ^2 in the whole strip of analyticity. Details will be given elsewhere [2b].

Final remarks. (i) In the models, the universal behavior is valid in a region around $s=4m^2$ which shrinks to zero as $\lambda\to 0$ [in $\exp[-\mathrm{cst}/\lambda]$ in view of Eqs. (8),(9),(10)]: it is thus consistent with the fact that $T\equiv 0$ for the free field. In other words, universality appears as soon as the coupling is switched on.

(ii) All the present results can be alternatively obtained by another method (presented in [2] and already applied in the past to the related problem of bound states [13a]) which exhibits other interesting features. This method is based on the use of another type of irreducible kernel U (derived from B through an integral equation without singularity at threshold [13b]) which is the analogue for $d=3$ of Zimmermann's Kmatrix [14].

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